Complete band gaps for liquid surface waves propagating over a periodically drilled bottom

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A plane-wave expansion approach is developed to solve the mild-slope equation for liquid surface waves propagating over a bottom with periodic structures. Band structures are calculated for the bottom periodically drilled with the square or triangular lattice of holes. Complete band gaps are found for both lattices. Parameters that influence the formation of band gaps are discussed.

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I. INTRODUCTION

When propagating in periodic structures, classical waves are greatly modulated by the introduced periodicity [1]. As a result of multiple Bragg scatterings, wave propagation in the periodic structures is characterized by band structures. Between bands there may exist a band gap within which wave propagation is absolutely forbidden. In the last few years, the idea of the existence of band gaps has been extended to photonic crystals for electromagnetic waves [2–7] and to sonic crystals for elastic waves [8,9] provided that their constituent materials are periodically arranged.

Liquid surface waves are also modulated by the introduced periodicity if propagating in periodic structures. Band structures and band gaps can also exist for liquid surface waves. Recently, there have been some theoretical calculations on the band structures and the possibility of the existence of band gaps for liquid surface waves propagating in periodic structures [10–15]. No complete band gaps, however, were found in Refs. [10–14]. Experimental work has also been carried out to study surface liquid waves propagating over a periodically drilled bottom [12,16]. In Ref. [12], band structure was measured. No complete (for all propagating directions) band gaps were observed. Only partial band gaps along certain directions in the Brillouin zone were found. Bloch waves were clearly observed due to the Bragg resonance [12,16].

In this paper, the plane-wave expansion method is applied for solving the mild-slope equation (MSE) [17–22] in order to investigate band structures for liquid surface waves propagating over a periodically uneven bottom. The paper is organized as follows. In Sec. II, the plane-wave expansion approach is developed for solving the MSE. The calculated results and discussions are presented in Sec. III. Conclusions are given in Sec. IV.

II. PLANE-WAVE EXPANSION APPROACH TO THE MILD-SLOPE EQUATION

Liquid surface wave propagation over an uneven bottom is a classical hydrodynamics problem [19,20,23]. It is a complicated three-dimensional problem. In the case of step-wise bottoms, waves consist of both propagative and evanescent terms. When the bottom slope is mild or when we only consider the overall wave propagation, the velocity potential Φ can be approximated as

$$\Phi(x,y,z,t)\approx \mathrm{Re}\left\{\varphi(x,y)\frac{\cosh k(z+h)}{\cosh kh}\exp(-i\omega t)\right\}, \tag{1}$$

where h is the variable liquid depth and φ is the complex horizontal variation. For inviscid liquids, the angular frequency ω and the local wave number k are related by the following dispersion relation

$$\omega^2=gk(x,y)\tanh[k(x,y)h(x,y)], \tag{2}$$

where g is the gravity's acceleration. It should be mentioned that the above slowly spatially varying dispersion relation is a consequence of a slowly varying depth function and is essentially a classical WKB approximation [24].

The MSE, originally proposed by Berkhoff [17,18], has been widely used to deal with the evolution of liquid surface waves over varying topography. The previous calculations of band structures for liquid surface waves are based either on the shallow water equations [11,12] or on the Helmholtz equation [13]. The MSE reduces to the Helmholtz equation in deep liquid and constant liquid depth, and to the shallow water equations in shallower liquid under the condition $kh \ll 1$. In the case of a bottom with periodic structures, we start with the MSE, which has wider applications and gives more reliable results than the shallow water equations [23,25]. The MSE may be taken in the form [21,22]

$$(\boldsymbol{\nabla}\cdot cc_g\boldsymbol{\nabla}+k^2cc_g)\varphi=0, \tag{3}$$

where $\boldsymbol{\nabla}\equiv(\partial_x,\partial_y)$ is the horizontal gradient, $c=\omega/k$ is the phase velocity, and $c_g=d\omega/dk$ is the group velocity. Although the theory is appropriate for slowly varying bottoms, the analysis and the qualitative physical features for drilled, discontinuous bottoms are expected to be not too different [14]. By introducing a new parameter

$$u\equiv cc_g/g=\frac{\tanh(kh)}{2k}\left(1+\frac{2kh}{\sinh(2kh)}\right), \tag{4}$$

the MSE can then be rewritten as the form

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$$(\nabla \cdot u \nabla + k^2 u)\varphi = 0. \tag{5}$$

Note that h, k,c, c_g, and u are all functions of horizontal coordinates $\mathbf{r}=(x,y)$. Reasons for introducing the new parameter u are that it has a length dimension and is an important parameter for the formation of band gaps (to be shown later). The physical meaning of u can be seen from the following discussions. Comparing Eq. (5) with the master equation for electromagnetic waves propagating in photonic crystals [5], it is easy to find that the introduced new quantity u plays a role similar to the dielectric constant ε in the master equation, a crucial parameter to determine photonic band gaps in photonic crystals. It should be noted that u is frequency dependent. For low frequencies, u is approximately the nominal liquid depth h. For high frequencies, however, u is proportional to ω^{-2}, independent of h. No band gaps are expected for high frequencies since these high frequency waves view the uneven bottom as even one. Therefore, u can be viewed as the *effective* liquid depth, an important parameter to determine the existence of band gaps.

We consider liquid surface waves propagating over a 2D periodically uneven bottom. A plane wave approach is adopted to solve the MSE, similar to electronic [26] and electromagnetic [5,27] waves propagating in periodic systems. In this system the liquid depth is a spatially periodic function. For a given frequency, k and u are different in the drilled and nondrilled area. They are also periodic functions of (x,y). The horizontal velocity potential φ must be the Bloch function. This Bloch function contains a Bloch wave vector due to the result of the introduced periodicity. The periodic functions u and k^2u can be expanded by plane waves

$$u(\mathbf{r}) = \sum_{\mathbf{G}} A_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{r}}, \tag{6}$$

$$k(\mathbf{r})^2 u(\mathbf{r}) = \sum_{\mathbf{G}} B_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{r}}, \tag{7}$$

where $\mathbf{G}=(G_x,G_y)$ is the 2D reciprocal lattice vectors. The field φ, which is the Bloch function, can be also expanded by plane waves, namely

$$\varphi(\mathbf{r}) = \sum_{\mathbf{G}} C_{\mathbf{G},\mathbf{q}} e^{i(\mathbf{G}+\mathbf{q})\cdot\mathbf{r}}, \tag{8}$$

where $\mathbf{q}$ is the Bloch wave vector in the first Brillouin zone [28]. The Fourier coefficients $A_{\mathbf{G}}$ and $B_{\mathbf{G}}$ can be obtained from

$$A_{\mathbf{G}} = \frac{1}{\Omega}\int_{\text{unit cell}} u(\mathbf{r}) e^{-i\mathbf{G}\cdot\mathbf{r}} d\mathbf{r}, \tag{9}$$

$$B_{\mathbf{G}} = \frac{1}{\Omega}\int_{\text{unit cell}} k(\mathbf{r})^2 u(\mathbf{r}) e^{-i\mathbf{G}\cdot\mathbf{r}} d\mathbf{r}, \tag{10}$$

where Ω is the area of the unit cell. Substituting Eqs. (6)–(8) into Eq. (5), we obtain

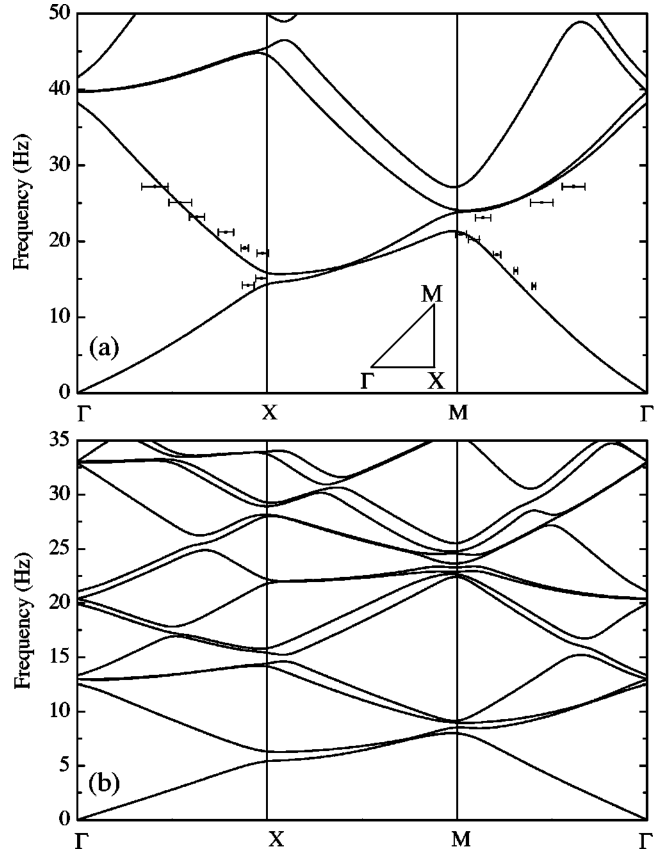

FIG. 1. Calculated band structures for the square lattice. Experimental results from Torres *et al.* [12] are shown with their corresponding error bars. The irreducible Brillouin zone is shown as inset. (a) $a=2.5$ mm, $R=0.75$ mm, $h_0=0.2$ mm, and $h_1=2.2$ mm. (b) $a=7.5$ mm, $R=1.75$ mm, $h_0=0.55$ mm, and $h_1=2.55$ mm.

$$\sum_{\mathbf{G}} Q_{\mathbf{G}',\mathbf{G}}(\mathbf{q},\omega) C_{\mathbf{G},\mathbf{q}} = 0, \tag{11}$$

where

$$Q_{\mathbf{G}',\mathbf{G}}(\mathbf{q},\omega) = [(\mathbf{G}+\mathbf{q})\cdot(\mathbf{G}'+\mathbf{q})]A_{\mathbf{G}'-\mathbf{G}} - B_{\mathbf{G}'-\mathbf{G}}. \tag{12}$$

It is easy to show that the matrix $\mathbf{Q}(\mathbf{q},\omega)$ is Hermitian. To ensure that Eq. (11) has nontrivial solutions, the determinant of the matrix $\mathbf{Q}(\mathbf{q},\omega)$ must be zero

$$\det|\mathbf{Q}(\mathbf{q},\omega)| = 0. \tag{13}$$

Band structures can then be obtained by solving the above equation. The number of plane waves is determined by the truncation of reciprocal lattice vectors. To guarantee the satisfactory convergence of the eigen values, enough plane waves are needed. It is noticed that Eq. (11) is not a standard eigenvalue problem since the expansion coefficients $A_{\mathbf{G}}$ and $B_{\mathbf{G}}$ are dependent on ω. A root finder is used to obtain the eigenvalues for a given wave vector $\mathbf{q}$.

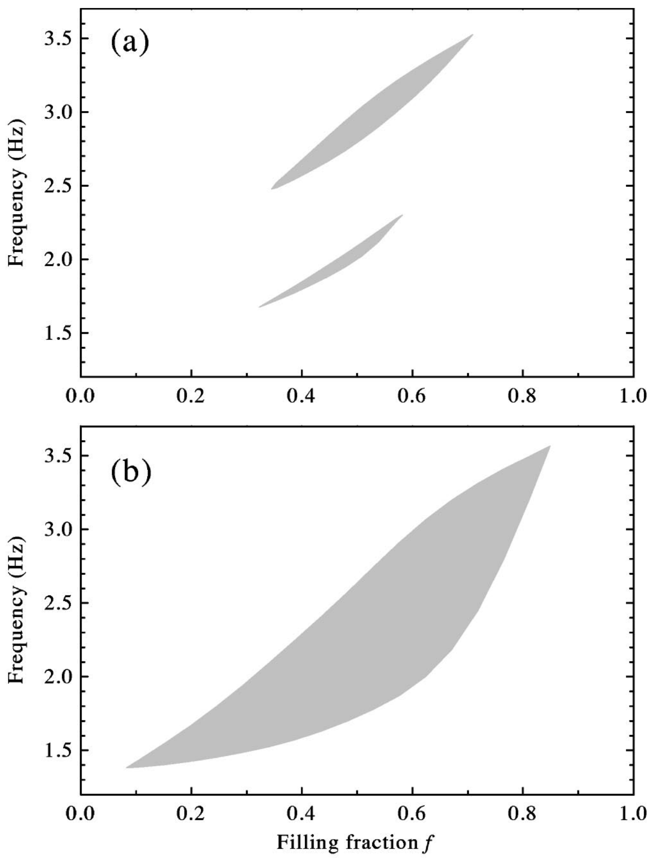

FIG. 2. Gap map as a function of the filling fraction for the square (a) and triangular (b) lattices of holes. Gray areas denote band gap regions. For both lattices, $a=22$ mm, $h_0=0.2$ mm, and $h_1=2.2$ mm.

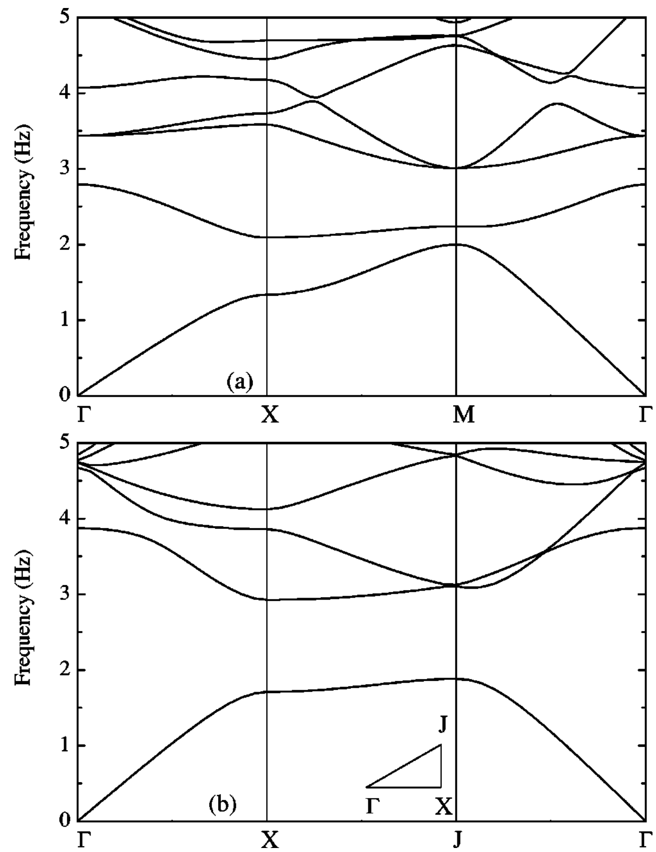

FIG. 3. Band structures for the square (a) and triangular (b) lattices. The parameters a, h_0, h_1 are the same as in Fig. 2 and the radius of the drilled holes is $R=8.8$ mm for both lattices. The irreducible Brillouin zone is shown as inset.

III. CALCULATED RESULTS AND DISCUSSIONS

We consider the bottom consisting of drilled cylindrical holes with radius R arranged in a periodic way. The liquid depth over the bottom is denoted by h_0 and the depth over the cylindrical holes by h_1. Therefore, h_1-h_0 is the depth of the cylindrical dimples. Two types of lattices of cylindrical holes are considered, i.e., the square lattice and the triangular lattice. The lattice constant is denoted by a. In all the following calculations, the results are obtained by using 253 plane waves for the square lattice and 285 plane waves for the triangular one. The accuracy is carefully checked by using plane waves of more than a thousand. Thus, we believe that all results presented here are accurate to within at least 1% of their true values. To compare with the experimental results, the liquid considered in our calculations is the same as in experiments [16,12]. It should be mentioned that for the geometries discussed below, the capillary effect will play a role [12] and should be taken into account. Therefore, the dispersion relation of Eq. (2) should be replaced by $\omega^2=g\kappa(1+d_c^2\kappa^2)\tanh(\kappa h)$ [16], where d_c is the capillary length, taken to be 0.93 mm [12].

In Fig. 1 the calculated band structures for the bottom drilled with the square lattice of holes together with the experimental results are shown. The geometry parameters for Fig. 1(a) are the same as in experiment [12] for studying the band structure, while those for Fig. 1(b) are the same as in experiment [16] for visualizing Bloch waves and domain walls. It can be seen from Fig. 1(a) that the calculated results are in good agreement with the experimental ones. For the geometry in Fig. 1(b), the Bloch wave patterns were presented in the experiment [16]. By inspecting the profiles of the corresponding modes, it is found that the calculated profiles are in accord with the observed ones.

Recall that in experiment [12] no complete band gaps were found. To investigate the possibility of the existence of band gaps, the gap maps as the function of the filling fraction f, defined as the fraction area occupied by the drilled holes in a unit cell, are given in Fig. 2 for the square and triangular lattices. For both lattices, $a=22$ mm, $h_0=0.2$ mm, and $h_1=2.2$ mm are taken. For the square lattice, there may exist two band gap regions. The first exists if the filling fraction is within the range between 0.32 and 0.58, while the second one exists for the filling fraction within the range between 0.35 and 0.71. The range of the filling fraction that renders the existence of the band gap possible for the first band gap is smaller than that for the second one. Moreover, the width of the second band gap is larger than that of the first one for the same filling fraction. For the filling fraction outside this range there are no band gaps. For the first band gap the optimal filling fraction that gives a maximum ratio of the gap width to the midgap frequency $\triangle\omega/\omega=4.6\%$ occurs at f

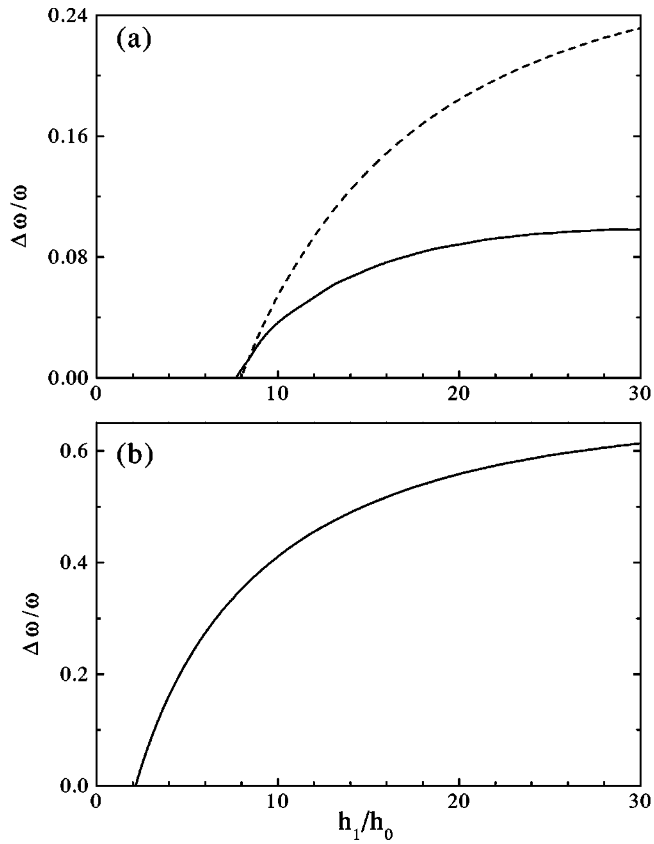

FIG. 4. Ratio of the gap width to the midgap frequency as a function of h_1/h_0 for the square (a) and triangular (b) lattices. Solid and dashed lines denote the first and second band gap, respectively. For both lattices, $h_0 = 0.2$ mm, $a = 22$ mm, and $R = 8.8$ mm.

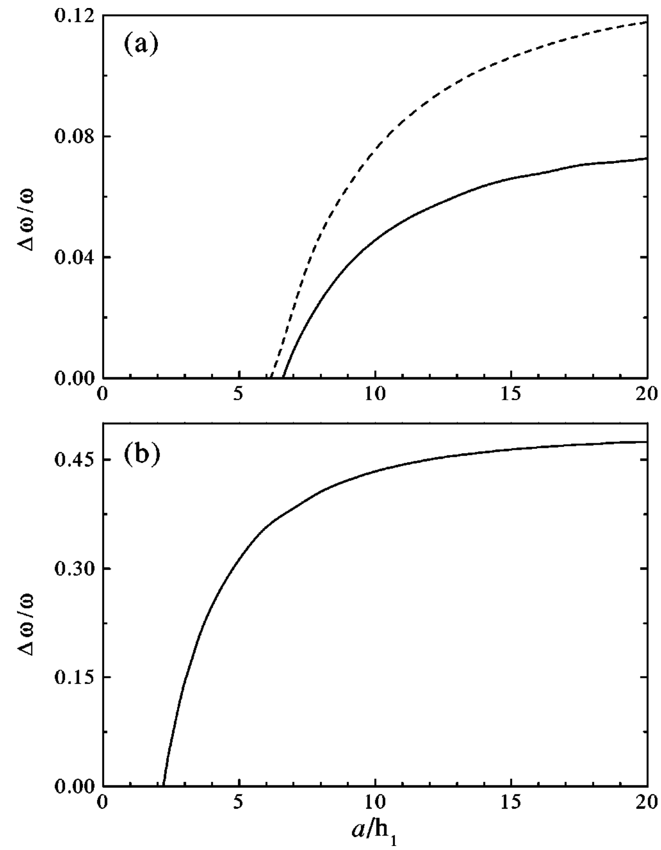

FIG. 5. Ratio of the gap width to the midgap frequency at the optimal filling fraction as a function of a/h_1 for the square (a) and triangular (b) lattices with $h_0 = 0.2$ mm and $h_1 = 2.2$ mm. Solid and dashed lines denote the first and second band gap, respectively.

$= 0.49$. For the second band gap the optimal filling fraction is $f = 0.51$, leading to a maximum $\triangle\omega/\omega = 7.5\%$.

For the triangular lattice, the first band gap exists for the filling fraction within the range between 0.08 and 0.85, which is much larger than that for the square lattice. The maximum value of the ratio of the gap width to the midgap frequency occurs at the optimal filling fraction $f = 0.58$, being 43%. The width of the band gap for the triangular lattice is much larger than that for the square lattice for the same filling fraction. Therefore, the triangular lattice is much easier to open up a band gap than the square lattice.

In Fig. 3, the band structures for liquid surface waves propagating over the periodically drilled bottom are shown. For both lattices, a, h_0, and h_1 are the same as in Fig. 2 and the radius of the drilled holes is $R = 8.8$ mm. For the square lattice, there are two band gaps. The first gap spans from 2.00 to 2.09 Hz, while the second one from 2.79 to 3.01 Hz. The width of the second band gap is larger than that of the first one. For the triangular lattice, the band gap is very large, spanning from 1.88 to 2.93 Hz. Clearly, there do exist band gaps for liquid surface waves propagating over the 2D periodically drilled bottom, contrary to the previous studies [12].

Basically, there are four parameters that influence the formation of band gaps, namely h_1/h_0, a/h_1, f, and the lattice symmetry. It is rather intuitive that h_1/h_0 is very crucial for the formation of a band gap. If it is too small there should be no band gaps. The ratio of the gap width to the midgap frequency as a function of h_1/h_0 is shown in Fig. 4 with $h_0 = 0.2$ mm and $a = 22$ mm for both lattices. For the square lattice, the minimum value of h_1/h_0 required for opening up a band gap is 7.70 for the first band gap and 7.96 for the second one. For the triangular lattice, the minimum value of $h_1/h_0 = 2.15$ is required to open up a band gap. For the same h_0 and a/h_1, the required minimum value of h_1/h_0 for opening up a band gap is much smaller for the triangular lattice than for the square lattice. This indicates again that the triangular lattice is more amiable for the formation of a band gap than the square lattice. It is noted that the ratio of the gap width to the midgap frequency increases with increasing h_1/h_0 and turns to a constant for large h_1/h_0 for h_1/h_0 larger than the minimum value required for opening up a band gap. This implies that the band structure is independent of h_1 for large h_1/h_0.

As regard to a/h_1, it is also an important parameter for the formation of a band gap. Its importance is not as intuitive as h_1/h_0 and is not noticed in the previous studies [11,12]. The ratio of the gap width to the midgap frequency as a function of a/h_1 at the optimal filling fraction is given in Fig. 5. There is no band gap for a/h_1 below a certain value. For the square lattice, the minimum a/h_1 required for producing a band gap is 6.60 and 6.16 for the first and second band gap, respectively. For the triangular lattice, this minimum value is

2.21. In the previous studies of the band structures for liquid surface waves, no complete band gaps were found [11,12]. The reason lies in the fact that a/h_1 is below the minimum required value for opening up a band gap.

For a/h_1 larger than the minimum value required for opening up a band gap, the ratio of the gap width to the midgap frequency increases with a/h_1, while the midgap frequency decreases accordingly. It is interesting to note that for large a/h_1 this ratio turns to a constant. Meanwhile, the optimal filling fraction also approaches a constant. This indicates a very interesting feature of scaling, similar to that in dielectric photonic crystals [5]. In other words, if we change the lattice constant scale of the system by a factor s and keep h_0, h_1/h_0, and f unchanged, the band structures are all scaled by the same factor, namely $\omega(\mathbf{q})/s$. This is because for very large a/h_1 the lower band gaps occur at low frequency. The effective depth u is approximately the nominal depth for low frequencies. This scaling property is checked by our numerical results. It should be noted that there is no such a scaling property for small a/h_1 and for high frequencies.

In the above discussions, we have used MSE in the study of band structures for liquid surface waves propagation over periodically drilled bottoms. In the previous theoretical and experimental studies, no complete band gaps were found. The reason lies in the geometry parameters chosen. If proper geometry parameters are used, complete band gaps can be found. If the shallow water equation is used instead of MSE, the band gaps found are usually larger than those predicted by MSE. This is because in the shallow water equation, the approximation $hk \ll 1$ is used. It leads to an effective liquid depth u being the nominal one even for high frequencies. In some cases, the shallow water equation predicts unphysical results because of the too rough approximation used.

IV. CONCLUSIONS

A plane-wave expansion method was developed to solve the MSE for surface liquid waves propagating over a bottom with periodic structures. Band structures were calculated for both the square and triangular lattices. It was found that for both lattices there do exist complete band gaps, contrary to the previous studies. It is more amiable to open up a band gap for the triangular lattice than for the square lattice. Moreover, the triangular lattice possesses a larger band gap than the square lattice. The parameters that influence the formation of band gaps were discussed. The reason why no band gaps were found in the previous studies lies in the fact that the ratio of the lattice constant to the depth over the drilled holes is below the required value. An interesting scaling property was found for low frequencies and a large ratio of the lattice constant to the depth over the drilled holes.

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